Introduction to error calculus

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Error calculus is the question of finding the error on a function of what is being measured. We study in this monograph error calculus with a mathematical point of view and in relationship with the experiment. Gauss was the first to propose an error calculus in the early 19th century. This calculation has a property of consistency makes it superior in many questions to other formulations proposed thereafter.

Let us consider the problem further in detail. The objectives of estimating the error can be of various kinds: a) Either one is interested in a specified situation and looking for a way to have *a pessimistic estimate of the error* allowing to speak in security. One is led then to work with bounds, intervals or domains, and see how they are transformed though calculations. This approach is ideal in theory, however, is terribly complicated or impractical in many cases. Even in the case of errors due to representations of real numbers in computer science this approach can only be carried out in very simple contexts. b) Or one does want just an order of magnitude of the error, easily obtained. This is often the case in engineering. In this context, a large number of practices exist. Since they must be simple and fast, they cannot make a real calculation of probabilities that describe how the probability of measurement is transmitted through the functions of the model. These calculations of "image distributions" or "image laws" are also quickly intractable. So often merely hybrid estimates are used, making for example the assumption that the measurement errors are Gaussian and calculating the transmission error by linearization in the neighborhood of the mean value. This makes services but we feel here some discomfort because ultimately at the end of the calculation we do not know exactly what the error obtained really is. c) Finally, we can look for the right mathematical language to make a true "limited expansion" of the error that is asymptotically exact if the error is small. It is then very close to a sensitivity analysis. But it is more precisely a *probabilistic* sensitivity analysis, which is important in the case of non-linear models.

The calculus proposed by Gauss belongs to the approach c) but only in one aspect, the simplest one on "infinitesimal variances" in the finite dimensional case. With this basic formalism it cannot be considered errors on functions or geometric objects such as surfaces, or on random processes. But the ideas of Gauss can be pushed further by an extension principle based on the theory of Dirichlet forms. We obtain then a complete Lipschitzian calculus that behaves perfectly by images and product and allows an easy construction of the basic notions of what is called (in stochastic analysis and finance) *Malliavin calculus*. It connects to the statistics by means of the Fisher information. We will only touch upon this theory giving references for the reader who wishes to deepen.

However, we will give a consequence of this theory. It provides an explanation for the delicate question of errors permanency raised by Henri Poincaré.

After an overview of the ideas of Gauss on the law of errors, those of Poincare on errors permanency, and the presentation of the Gauss calculus and its coherence, we will present the extension tool that allows to build a Lipschitz calculus and its axiomatization. We then discuss the relationship with experiments and statistics and give examples of finite and infinite dimensional repeated samples where appears the phenomenon of the permanence of errors. This text is based on articles [14] and the book [15].

Instead of discrete variables, continuous variables are often flawed by errors. To tackle this problem several attitudes are encountered. We propose here to speak of errors rigorously, supposing them small and controlling the terms of their expansion in the calculations. This approach was initiated by Legendre, Laplace and Gauss in the early 19th century, in a series of works denoted by *Classical Error Theory*. The most famous of them is the demonstration by Gauss of the "law of errors" in which he shows, with some implicit assumptions that will be highlighted by other authors, that if we consider in an experimental situation, that the arithmetic mean of measurements is the best value to take into account, we must admit that the errors follow a normal distribution. His argument is probabilistic : the quantity to be measured is a random variable X and the measurements $X_1, \ldots, X_n$ are supposed to be conditionally independent knowing X.

At the end of the same century, in his course *Calcul des probabilités* Henri Poincaré returns to this question by showing that if we weaken some assumptions of Gauss, laws other than the normal distribution can be obtained. He discusses at length a new and delicate point : the phenomenon of permanence of errors, that he explains in the following way :

“Avec un mètre divisé en millimètres, on ne pourra jamais, écrit-il, si souvent qu’on répète les mesures, déterminer une longueur à un millionième de millimètre près”. [With a meter graduated in millimeters, it will never be possible, so often measurements are repeated, to determine a length up to a millionth of millimeter.]

This phenomenon is well known to physicists, in the history of physics we have never been able to make accurate measurements with crude instruments, see [1]. That means doing a lot of measurements and take the average is not enough to guarantee an arbitrarily fine precision. We will explore this question and give a mathematical explanation of this phenomenon. Poincaré did not develop a mathematical formalism for this, he insists, however, on the advantage of assuming small errors because then the argument of Gauss becomes compatible with non-linear changes of variables and can be written though a differential calculus.

1. The error calculus of Gauss

Twelve years after his demonstration justifying the normal law, Gauss is interested in the propagation of errors. (*Theoria combinationis* 1821). Given a quantity $U = F(V_1, V_2, \ldots)$ function of other quantities $V_1, V_2, \ldots$, he poses the problem of calculating the quadratic error on U knowing the quadratic errors $\sigma_1^2, \sigma_2^2, \cdots$ on $V_1, V_2, \ldots$, these errors being supposed small and independent.
His answer is the following

$$\sigma_U^2 = (\frac{\partial U}{\partial V_1})^2 \sigma_1^2 + (\frac{\partial U}{\partial V_2})^2 \sigma_2^2 + \cdots \tag{1}$$

and he gives also the covariance of the error on U and another function of $V_1, V_2, \ldots$.
Formula (1) possesses a property that gives it a great superiority with respect to other formulae often proposed in textbooks. It is the *coherence property*. With a formula such that

$$\sigma_U = \left|\frac{\partial U}{\partial V_1}\right| \sigma_1 + \left|\frac{\partial U}{\partial V_2}\right| \sigma_2 + \cdots \tag{2}$$

errors may depend on the way the function F is written. Already in dimension 2 if we apply (2) to a linear one to one map and then to its inverse, we obtain that the identity map increases the errors what is unacceptable.
This doesn't happen with Gauss calculus. In order to see that, let us introduce the differential operator

$$L = \tfrac{1}{2}\sigma_1^2 \frac{\partial^2}{\partial V_1^2} + \tfrac{1}{2}\sigma_2^2 \frac{\partial^2}{\partial V_2^2} + \cdots$$

and let us remark that (1) writes

$$\sigma_U^2 = LF^2 - 2FLF$$

Coherence comes then from the coherence of the transport of a differential operator by a function : If L is such an operator, if u and v denote regular one to one maps and if $\theta_u L$ denotes the operator

$$\varphi \mapsto L(\varphi \circ u) \circ u^{-1}$$

we have

$$\theta_{v \circ u} L = \theta_v(\theta_u L).$$

Now the errors on $V_1, V_2, \ldots$ may be non independent and may depend on the values of $V_1, V_2, \ldots$: we consider a field of positive symmetric matrices $(\sigma_{ij}(v_1, v_2, \cdots))$ on $\mathbf{R}^d$ representing the conditional variances and covariances of the errors knowing the values $v_1, v_2, \ldots$ of $V_1, V_2, \ldots$ and the calculus writes :

$$\sigma_F^2 = \sum_{ij} \frac{\partial F}{\partial V_i}(v_1, v_2, \cdots) \frac{\partial F}{\partial V_j}(v_1, v_2, \cdots) \sigma_{ij}(v_1, v_2, \cdots) \tag{3}$$

The error calculus of Gauss deals with variances and covariances of errors regardless of the mean error that is to say without taking in account the biases. It is important to emphasize that this is why it involves only first derivatives. Indeed, if we start from a situation where the errors are centered, after a non linear transform errors are no more centered and it can be verified that the bias of the error is of the same order of magnitude as the variance (see [15]). If we apply other non linear regular applications this situation will persist. This allows to see that *the variances can be calculated by a first order differential calculus involving only the variances, while the means of errors need a differential calculation of the second order which involves the means and the variances.*

The coherence of the Gauss calculus allows it to be geometrized. If a quantity varies on a manifold, the error may be attached to the variety as a geometric object. The variance of the error is given by a quadratic form which is a Riemannian metric on the manifold. It is possible to take images by injective C^1 applications in a coherent formalism independent of the way the functions are written. This relates to the theory of diffusion processes on manifolds for which we refer to references [2].

2. Error calculus with extension tool

The calculus of Gauss is limited by the fact that it doesn't possess any extension tool. By an extension tool we mean, in mathematics, a mean of calculating on limit objects, i.e. defined by limits. Starting from the error on (V_1 , V_2 , V_3) the Gauss calculus allows to compute the error on a differentiable function of (V_1 , V_2 , V_3) and that's all.

In particular we would like to be able to compute the error when the function is not explicit but given by a differential equation or an integral or as solution of a boundary value problem. Also we would like to extend this calculus to Lipschitz functions since it is *a priori* clear that a function with Lipschitz constant ≤ 1, is contracting and therefore diminishes the errors.

Also in a frequent situation in probability theory where we have a sequence of quantities $X_1 , X_2 , \ldots ,$ $X_n , \ldots$ and where we know the errors on the regular functions of a finite number of the X_n , we would like to deduce the error on functions of an infinite number of the X_n or at least some of them. We give examples below.

It is actually possible to equip the error calculus with a natural extension tool.

For this we come back to the initial idea of Gauss to consider that erroneous quantities are random quantities, say defined on a probability space (Ω, A, $\mathbf{P}$). The quadratic error on a random variable X is itself random, we denote it $\Gamma[X\ ]$. It is supposed infinitesimal but this doesn't appear in the notation, as if we had an infinitesimal unit for errors fixed through the problem. The tool is the following one: we suppose that if $X_n \to X$ in $L^2(\Omega, A, \mathbf{P})$ and if the error $\Gamma[X_m - X_n\ ]$ on $X_m - X_n$ may be made as small as one wants in $L^1(\mathbf{P})$ for m, n large enough, then the error $\Gamma[X_n - X\ ]$ tends to zero in $L^1(\mathbf{P})$.

It is a reinforced coherence principle since it means that the quadratic error on X is attached to X as a mathematical mapping and that if the pair (X_n , quadratic error on X_n) converges in a suitable sense, it converges necessarily to (X, quadratic error on X).

This is axiomatized in the following way :
We call *error structure* a probability space equipped with a local Dirichlet form possessing a squared field operator.
It is a term

$$(\Omega, A, \mathbf{P}, \mathbf{D}, \Gamma)$$

where (Ω, A,$\mathbf{P}$) is a probability space, satisfying the four following properties :

(1.) $\mathbf{D}$ is a dense subvectorspace of $L^2(\Omega,A,\mathbf{P})$

(2.) Γ is a symmetric bilinear application of $\mathbf{D}$x$\mathbf{D}$ into $L^1(\mathbf{P})$ satisfying the functional calculus of class $C^1 \cap Lip$, what means that if $u \in \mathbf{D}^m$ and $v \in \mathbf{D}^n$, for F and G of class C^1 and Lipschitz from $\mathbf{R}^m$ [resp. $\mathbf{R}^n$] into $\mathbf{R}$, we have $F \circ u \in \mathbf{D}$ and $G \circ v \in \mathbf{D}$ and

$$\Gamma[F \circ u, G \circ v] = \sum_{ij} F_i'(u) G_j'(v) \Gamma[u_i, v_j] \qquad \mathbf{P}\text{-a.s.}$$

(3.) The bilinear form $\mathcal{E}[f,g] = \mathbf{E}[\Gamma[f,g]]$ is closed, what means that $\mathbf{D}$ is complete for the norm $\|\cdot\| = (\|\cdot\|^2_{L^2(P)} + E[.,.]\ )^{1/2}$.

(4.) Finally we suppose $1 \in \mathbf{D}$ et $\Gamma[1,1]=0$.

<u>Comment</u>. We always note $\mathcal{E}[f]$ for $\mathcal{E}[f,f]$ and also $\Gamma[f]$ pour $\Gamma[f,f]$. With this definition the form $\mathcal{E}$ is a Dirichlet form, notion introduced by Beurling and Deny [3] [13] as a tool in potential theory which received a probabilist interpretation in terms of symmetric Markov processes thanks to the works of Silverstein and Fukushima cf [3] [5] [12]. The operator Γ is the *squared field operator* also called the *carré du champ operator* associated to $\mathcal{E}$. It has been studied by many authors in more general frameworks than the present setting cf [4] [5].

2.1 First examples

a) A simple example of error structure is the term

$$(\mathbf{R}, B(\mathbf{R}), \mu, H^1(\mu), \gamma\ )$$

where μ is the reduced normal law $\mu = N(0, 1)$ and $H^1(\mu)$ is the set of $f \in L^2(\mu)$ such that $f\,'$ (in the sense of distributions) be in $L^2(\mu)$ with $\gamma\,[f] = f'^2$ for $f \in H^1(\mu)$. This structure is associated with the real valued Ornstein-Uhlenbeck process cf. [15].

b) Let D be a connected open set of $\mathbf{R}^d$ with unit volume, λ_d be the Lebesgue measure, we take $(\Omega, \mathcal{A}, \mathbf{P}) = (D, \mathcal{B}(D), \lambda_d)$. We define

$$\Gamma[u,v] = \sum_{ij} \frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_j} a_{ij} \qquad \text{for} \quad u,v \in C_K^\infty(D)$$

where the a_{ij} are applications from D into $\mathbf{R}$ such that

$$a_{ij} \in L^2_{loc}(D), \quad a_{ij} = a_{ji}, \quad \frac{\partial a_{ij}}{\partial x_k} \in L^2_{loc}(D), \quad \sum_{ij} a_{ij}(x)\xi_i\xi_j \geq 0, \quad \forall \xi \in R^d, \forall x \in D.$$

It is possible to show that the form $\mathcal{E}[u, v] = \mathbf{E}\Gamma[u, v]$ with $u,v \in C_K^\infty(D)$ is closable (cf [5]) in other words, there exists an extension of Γ to a subspace $\mathbf{D}$ of L^2, $\mathbf{D} \supset C_K^\infty(\mathbf{D})$ such that $(\Omega, \mathcal{A}, \mathbf{P}, \mathbf{D}, \Gamma)$ be an error structure.

2.2 Taking the image : error on a the result of a function of measured quantities

The image of an error structure by an application is defined very naturally and yields still an error structure, as soon as the application satisfies some rather weak assumptions cf [6]. In particular if $(\Omega, \mathcal{A}, \mathbf{P}, \mathbf{D}, \Gamma)$ is an error structure and if X is a random variable with values in $\mathbf{R}^d$ whose components are in $\mathbf{D}$, noting $\mathbf{P}_X$ the probability distribution of X and putting

$$D_X = \left\{ f \in L^2(P_X) : \quad f \circ X \in D \right\}$$

$$\Gamma_X[f](x) = E[\Gamma[f \circ X] \mid X = x], \qquad f \in D_X$$

then the term $(\mathbf{R}^d, \mathcal{B}(\mathbf{R}^d), \mathbf{P}_X, D_X, \Gamma_X)$ is an error structure.

2.3 Taking the product : error on a pair or a family of independent quantities

The product of two or of a countable infinity of error structures is always defined and yields an error structure. We obtained easily in this way error structures on infinite dimensional spaces, cf [6], for instance on the Wiener space or on the Poisson space and on models derived from them. It is a useful way of introducing the Malliavin calculus pedagogically, cf [7].

Let us show, as example, the construction of the Ornstein-Uhlenbeck structure on the space of the Brownian motion, i.e. on the Wiener space.

Let us come back to the one dimensional structure of example a)

$$(\mathbf{R}, \mathcal{B}(\mathbf{R}), \mu, H^1(\mu), \gamma)$$

and let us consider the infinite product :

$$(\Omega, \mathcal{A}, \mathbf{P}, \mathbf{D}, \Gamma) = (\mathbf{R}, \mathcal{B}(\mathbf{R}), \mu, H^1(\mu), \gamma)^{\mathbf{N}} = (\mathbf{R}^{\mathbf{N}}, \mathcal{B}(\mathbf{R}^{\mathbf{N}}), \mu^{\mathbf{N}}, \mathbf{D}, \Gamma).$$

The coordinate mappings X_n, by construction of the product, are reduced independent Gaussian, belong to $\mathbf{D}$ and satisfy

$$\Gamma[X_n] = 1$$

$$\Gamma[X_m, X_n] = 0 \quad m \neq n.$$

Let ξ_n be an orthonormal basis of $L^2(\mathbf{R}_+, dt)$. We put

$$B_t = \sum_{n=0}^{\infty} \int_0^t \xi_n(s)ds.X_n$$

Then $(B_t)_{t\geq 0}$ is a Brownian motion and if $f \in L^2(\mathbf{R}_+)$ writes $f = \Sigma_n a_n \xi_n$, the random variable $\Sigma_n a_n X_n$ is denoted $\int_0^\infty f(s)\,dB_s$ by extension of the case where f is a step function.

We have $\int f(s)dB_s \in D$ and

$$\Gamma[\int f(s)dB_s] = \Gamma[\sum_n a_n X_n] = \sum_n a_n^2 \Gamma[X_n] = \sum a_n^2 = \|f\|^2_{L^2}$$

and, by the extension tool, the error calculus extends to other Brownian functionals as solutions of stochastic differential equations with Lipschitz coefficients cf [6] [7] [11].

3. Error calculus and statistics

To switch from the Gauss error calculus to a Lipschitz complete calculus it is necessary to have a probability measure. If quantities vary but remain deterministic as often in mechanics, they must be placed in a probabilistic framework. It is the triplet $(\Omega, \mathcal{A}, \mathbf{P})$ of an error structure $(\Omega, \mathcal{A}, \mathbf{P}, \mathbf{D}, \Gamma)$.

One approach is to follow the ideas of E. Hopf in the 1930s who, in the spirit of the work of Poincaré, showed, using general forms of limit theorems in distribution, that many dynamical systems have natural laws of probability that can be taken as *a priori* distributions, see [8] and [15].

A second way consists to start with a second order elliptic operator L satisfying

$$\Gamma[F] = LF^2 - 2FLF \tag{4}$$

(whose only the second order terms are determined by this relation) which yields the framework of an error calculus for the variances and the biases. Then to construct the invariant probability measure with respect to which the diffusion of generator L is symmetric.

We will follow a third way which is more strongly connected with applications. Let us consider that the experimental conditions are sufficiently specified for the probability measure $\mathbf{P}$ be obtained as usually by statistical methods and we will show that the statistics yield also actually the operator Γ hence finally the error structure, at least on a minimal domain for Γ.

Let us consider an erroneous d-dimensional quantity X . The image space by X is

$$(\mathbf{R^d}, \mathcal{B}(\mathbf{R^d}), \mathbf{P}_X(dx))$$

The operator Γ that we try to define writes under the form

$$\Gamma_X[F](x) = \sum_{i,j=1}^{d} F_i'(x)F_j'(x)a_{ij}(x)$$

where the matrix $A(x) = (a_{ij}(x))$ is positive symmetric, it is this matrix that we need to know, it represent the accuracy of the knowledge of X at the point x.

Let us remark that is $G : \mathbf{R^d} \rightarrow \mathbf{R^m}$ is of class $C^1 \cap Lip$ by the functional calculus, the random variable $G(x)$ is known with the accuracy

$$\Gamma_X[G, G](x) = \nabla_x G.A(x).(\nabla_x G)^t \tag{5}$$

where $\nabla_x G$ is the Jacobian matrix of G at x.

Now in order to know X , under the conditional law $X = x$ denoted $\mathbf{E}_x$, we perform measurements that are *estimates* of the parameter x. Let T be such an estimate with values in $\mathbf{R}^m$ with covariance matrix

$$\mathbf{E}_x\,[(T - \mathbf{E}_x\,[T\,]).(T - \mathbf{E}_x\,[T\,])^t\,].$$

Under the statistical hypotheses called "of regular model" the Fréchet-Darmois-Cramer-Rao inequality writes

$$\mathbf{E}[(T - \mathbf{E}_x[T\,]).(T - \mathbf{E}_x[T\,])^t] \geq \nabla_x\mathbf{E}_x T\,.J\,(x)^{-1}.(\nabla_x\mathbf{E}_x T\,)^t \tag{6}$$

In the sense of the order of cone of the positive symmetric matrices, where $J\,(x)$ is the *Fisher information matrix*, cf [9]. The best precision that can be reached on X is therefore $J\,(x)^{-1}$ and comparison of (5) and (6) leads to put

$$\Gamma(x) = J\,(x)^{-1}.$$

It is easily seen that this definition is compatible with regular change of variables : if $\psi(x)$ is estimated instead of x, we obtain as error structure the image by ψ of the error structure of X.

This natural connection between Fisher information and the approach of errors based on Dirichlet forms opens a series of questions which are still at the level of research : a) Under which hypotheses can we obtain directly $J(x)^{-1}$ possibly singular without to take the inverse of the Fisher information matrix ? b) Do the asymptotic methods of statistics give tools for studying the closability of Dirichlet pre-forms on R^d ? See [10].

4. What happens when we repeat the measurements ?

The introduction of errors operators in addition to the language of probability theory allows to treat with more finesse the question of repeated samples and to answer by explicit models to the phenomenon of error permanency emphasized by Poincaré.

As we shall see some projective systems for which the limit of the probability spaces exists, do not possess a limit error structure, but define only an error pre-structure in the following sense :

A term $(\Omega, \mathcal{A}, \mathbf{P}, \mathbf{D}^0, \Gamma)$ is an error pre-structure if $(\Omega, \mathcal{A}, \mathbf{P})$ is a probability space and if $\mathbf{D}^0$, Γ satisfy properties (1.), (2.) and (4.) of error structures but not necessarily property (3.).

There are therefore closable and non closable error pre-structures. Images and products are defined easily for error pre-structures. Let us fix some notation for projective systems under usual hypotheses cf [6] :

Given measurable spaces (E_i, F_i) *i integer*, for α in the set J of finite parts of de $\mathbf{N}^*$ ($=\mathbf{N}\backslash\{0\}$) a projective system of error structures (or of error pre-structures) ia family $(E_\alpha, F_\alpha, m_\alpha, \mathbf{D}^0_\alpha, \Gamma_\alpha)$ of error (pre-)structures where $(E_\alpha, F_\alpha) = \prod_{i\in\alpha} (E_i, F_i)$ are compatible in usual sense. Putting then $D^0 = \bigcup_{\alpha\in J} D^0{}_\alpha$ this defines an error pre-structure :

$$(E, F, m, \mathbf{D}^0, \Gamma)$$

whose projections are the $(E_\alpha, F_\alpha, m_\alpha, \mathbf{D}^0_\alpha, \Gamma_\alpha)$.

The projective systems that we consider from now on are such that the (E_i, F_i) are all identical and such that the projective system be auto-isomorphic by translation of the indices, this hypothesis is taken in order to represent repeated samples.

We shall give three examples. In the examples 4.1 and 4.2 the situation whose repeated samples are considered is finite dimensional, it is a finite dimensional probabilistic model with erroneous quantities. The asymptotic properties of repeated samples are different in cases 4.1 and 4.2. In the example 4.3 the probabilistic model whose repeated samples are studied is a space of stochastic processes, infinite dimensional, this case is important because it gives the idea of the most interesting applications (statistical physics, filtering and forecasting, mathematical finance).

4.1 Independent samples, correlated errors, case of asymptotic convergence

In this first example we take, the errors are correlated, (as suggested Poincaré) and the projective system is closable :

$$(E_i, F_i) = ([0, 1], B([0, 1])) \qquad \forall i \in \mathbf{N}^*$$

The pre-structure $(E_\alpha, F_\alpha, m_\alpha, \mathrm{D}^0_\alpha, \Gamma_\alpha)$ is defined by

$$(E_\alpha, F_\alpha, m_\alpha) = ([0, 1]^{|\alpha|}, B([0, 1])^{|\alpha|}, \lambda_{|\alpha|})$$

where

$|\alpha| = \mathrm{card}(\alpha)$ and $\lambda_{|\alpha|}$ is the Lebesgue measure of dimension $|\alpha|$, one takes $\mathbf{D}^0_\alpha = C^0_K(]0, 1[^{|\alpha|})\otimes\mathrm{R}$, and for $u,v \in D^0_\alpha$ we put $\Gamma_\alpha[u,v] = \sum_{i,j\in\alpha} \frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_j} a_{ij}$ where the a_{ij} are constant and such that the matrices $(a_{ij})_{i,j\in\alpha}$ be positive symmetric.

In this case it is possible to show that the pre-structures $(E_\alpha, F_\alpha, m_\alpha, \mathrm{D}^0_\alpha, \Gamma_\alpha)$ are closable and also the pre-structure defined by their projective system. Its closure difines an error structure

$$(E, F, m, \mathbf{D}, \Gamma)$$

whose projections are the closures of the $(E_\alpha, F_\alpha, m_\alpha, \mathrm{D}^0_\alpha, \Gamma_\alpha)$.

The properties of this structure $(E, F, m, \mathbf{D}, \Gamma)$ will give the answer to the question of Poincaré. We are in the case where the samples are independent but the errors correlated. If $a_{ij}=1$, it may be shown by the law of large numbers that *the error on the mean vanishes*.

This is still the case if $a_{ij}=a(i-j)$ where a is a function such that $\Sigma\xi_i\xi_j a(i-j)>0$, thanks to the theorem of representation of Bochner cf [14].

It is no more the case in the following example.

4.2 Independent samples, correlated errors, no asymptotic convergence

Let us suppose that each measurement yields a scalar quantity, we consider the product probability structure representing the independent repeated samples X_i (coordinate mappings) and on this product space we define on regular functions the squared field operator (operator representing the les variances and co-variances of the errors) by

$$\Gamma_\alpha[u] = \left(\sum_{i\in\alpha} u'_i f(X_i)\right)^2 + \sum_{i\in\alpha} u'^2_i . g(X_i)$$

A suitable choice of functions f and g allows to show that the time average of the samples tends indeed to their expectation, but the error on the mean, that is to say :

$$\Gamma\left[\tfrac{1}{N}\sum_{n=1}^{N} h(X_n)\right]$$

tends to $\left(\int h'.f\right)^2$ that is not zero generally, cf [14] (2001) example B.

This model describes a situation similar to the one quoted by Poincaré where doing the mean of a large number of samples do not makes the error tend to zero.

4.3 Indications on infinite dimensional cases and Malliavin calculus

As an example involving the infinite dimension and allowing to show the formalism and the arguments, let us consider the case of a twine of length L thrown on a plan whose the total length of the projection on the axis Ox, is measured for instance by means of thin parallel lines.

By repeated samples this allows to measure the length of the twine as we shall see in a moment. This can be modelled in the following way : the twine is parametrized by

$$X(t) = X_0 + \int_0^t \cos(\varphi + B_s)ds$$

$$Y(t) = X_0 + \int_0^t \sin(\varphi + B_s)ds \qquad 0 \le t \le L \le 1$$

where B is a standard Brownian motion and ϕ uniform on the circle, independent of B. We measure the quantity

$$A(\varphi,\omega) = \int_0^L \left|\cos(\varphi + B_s)\right| ds$$

The expectation $\mathbf{E}A$ is obtained by repeated samples and the length L of the twine is given by the formula

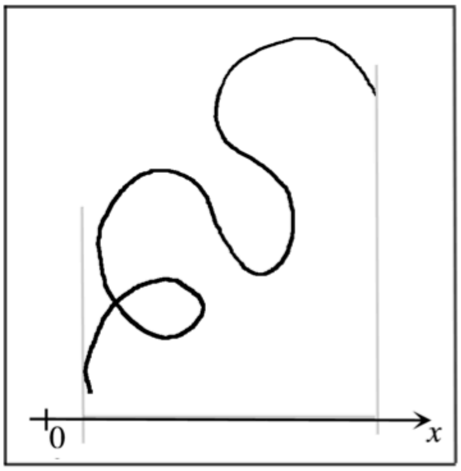
$$\mathrm{E}A = \frac{2L}{\pi}$$

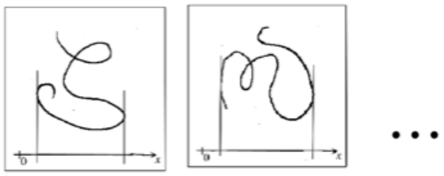
that comes immediately from the expression of A by integration since ϕ and B are independent.

As hypothesis on the errors we suppose that there is an error on ϕ and an error on B independent but that the errors on the different samples are correlated.

On ϕ we consider an error similar to that of the case 4.2.

On B we consider for simplicity the error given by the Dirichlet form associated with the Ornstein-Uhlenbeck semi-group cf [7] whose construction was given previously.





Measure of the length of a twine by its projection on Ox

The question we address is to estimate the error of this method. We understand that the calculus à la Gauss is not sufficient here and that we need a more elaborated mathematical framework. This formalism (Dirichlet forms) shows as well the hypotheses that must be explicited.

Explicit computations may be performed if the operator that describes the correlation of the errors is specified. For instance if we take the multiplication by a function $a(\varphi,\omega)$ for the Wiener space and by $b(\varphi,\omega)$ for the angle φ, with $a(\varphi,\omega)=b(\varphi,\omega)\sqrt{g}\,(\varphi)=1_{[0,2\pi]}(\varphi)$, we obtain for the quadratic asymptotic error

$$\lim_{N\uparrow\infty}\Gamma\Big[\tfrac{1}{N}\sum_{n=1}^{N}A_n\Big] = \int_{[0,L]}\Big[\tfrac{1}{2\pi}\Big(\int_R |\cos x| - |\sin x| dx\Big)\int_t^L \frac{e^{-\frac{x^2}{2s}}}{\sqrt{2\pi s}}\,ds\,dx\Big]^2 (dt + \delta_0(t))$$

expression that shows that for infinite dimensional phenomena, the results of error calculations depend highly on the choice of the hypotheses concerning the probability distributions and the correlations.

Here the twine is of class C^1. Other models may be studied and computed easily when the twine is supposed to be C^2, etc. voir [15].

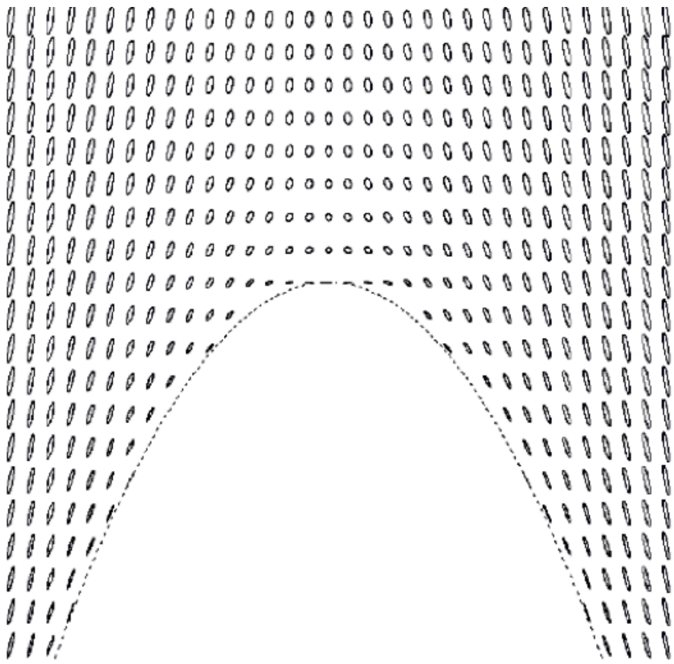
Conclusion

In this introductory text we wanted to convince the reader of the interest to push forward the ideas of Gauss on an error calculus based on differential calculus what gives it some likeness with sensitivity calculus.

1) It is not, however, a sensitivity analysis in the sense of a derivation with respect to a parameter of the model, a strictly deterministic operation. Because the random nature of the error (even if it is infinitely small) makes that after a non-linear mapping, the average error is not the image of the average of the initial error. It is therefore a probabilistic sensitivity analysis. And this translates into a calculus of second order for the biases.

2) Why use Dirichlet forms? They are a mathematical concept, in my opinion, as important as the concept of probability space. Their properties are also often similar to those encountered in probability.

I think I have demonstrated mathematically that this is the natural object which is required in the calculation of errors as soon one wishes to reason about less basic physical objects. Cf "When and how an error yields a Dirichlet form" *Journal of Functional Analysis* Vol 240, Issue 2 , (2006) 445-494.



Example of elliptic field obtained as errors
On a pair of quantities depending on an erroneous Brownian motion

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