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The impact of randomness on the distribution of wealth: Some economic aspects of the Wright-Fisher diffusion process

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Abstract

In this paper we consider some elementary and fair zero-sum games of chance to study the impact of random effects on the wealth distribution of N interacting players. Even if an exhaustive analytical study of such games between many players may be tricky, numerical experiments highlight interesting asymptotic properties. From a mathematical perspective, we interestingly recover for small and high-frequency transactions some diffusion limits extensively used in population genetics. Finally, the impact of small tax rates on the preceding dynamics is discussed for several regulation mechanisms.

Keywords: Fair zero-sum games, Wright-Fisher diffusions, Impact of tax rate.

JEL classification: C32, C63, D31.

1 Introduction and numerical experiments

We consider in this paper an economy simplified to the extreme and reduced to random games between agents, the games being fair in expectation. This situation may be seen as a basic model of randomness in the physical world or modeling the effect of volatility on prices, considering that at least for the first order of magnitude the games may be considered with zero expectations. The focus is therefore on social and collective phenomena appearing in this purely speculative framework with or without regulation mechanisms.

The objective is to study the dynamic of some elementary Markov games of chance to highlight the impact of random effects on the wealth distribution of interacting players. Surprisingly, even if zero-sum games of chance (supposed to be fair in expectation) are played at any round, wealth distribution converges toward the maximal inequality case. This qualitative behavior has already been empirically observed (see [25] Chap. 15, [3]) and here the conclusion is both supported by numerical and mathematical arguments at the very least for small and high-frequency transactions where Wright-Fisher diffusion processes naturally appear as limit models. We also investigate the impact of some regulation mechanisms on the qualitative behavior of the considered models, in particular, our findings provide a very simple agent-based framework for understanding how the Beta distribution, widely used in the literature as descriptive models for the size distribution of income ([22], [4]), arises in wealth repartition problems.

Let us start by some elementary definitions and examples.

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1.1 Fair elementary zero-sum games of chance (FEG)

We consider two players playing during $n \in \mathbb{N}^*$ consecutive rounds a zero-sum game of chance[1]. If we denote by $(P_k^i)_{k\in\{1,...,n\}}$ the payoff process (defined on the probability space $(\Omega, \mathcal{A}, \mathbb{P})$) of the player $i \in \{1, 2\}$, starting from a constant initial wealth X_0^i, we have $P_k^1 = -P_k^2$ (zero-sum game) and the wealth process is given by

$$X_k^i = X_0^i + \sum_{1 \leqslant j \leqslant k} P_j^i.$$

Definition 1. *We say that the preceding zero-sum game of chance is fair in expectation on* $(\Omega, \mathcal{A}, \mathbb{P})$ *(and we will write* **FEG** *for fair elementary game) if* $\forall k \in \{1, ..., n\}$, $\mathbb{E}_{\mathbb{P}}[P_k^i] = 0$.

Remark: *Let us emphasize that considering games fair in expectation is philosophically the most natural if we have no additional reason for the presence of biases, and that it was indeed the first historical approach of the purse taken by Louis Bachelier in [2] who supposed that "the expectation of the speculator is zero".*

A game is fair in expectation as soon as the random variables P_k^i are symmetric (e.g. for the classical two players fair coin flipping game where each player wins or losses one unit), but this condition is not necessary (and not really economically relevant) as we are going to see in the three following one stage (n=1) examples:

- **Example 1: Calabash game.** (See [6], p. 57)

In its simplest form, this traditional African game also known as the gourd game is between two players. Each player uses seeds of a certain color and all the seeds have an identical form. At each turn, each player puts into the gourd as many seeds as he wants. The gourd is a sort of large hollowed-out melon where care has been taken to leave inside a stem on which a single seed can sit. The gourd is shaken until one of the seeds comes to rest on this stem and the player of the corresponding color collects all the seeds in the gourd. After this, players exchange the seeds so as to keep the same color. For the one stage game, if we denote by N_1^i the number of seeds bet by player i $(0 < N_1^i \leqslant X_0^i)$, we have

$$P_1^1 = N_1^2 \mathbf{1}_{U_1 \leqslant \frac{N_1^1}{N_1^1+N_1^2}} - N_1^1 \mathbf{1}_{U_1 > \frac{N_1^1}{N_1^1+N_1^2}}$$

where U_1 follows a uniform distribution on $[0, 1]$. The game is a FEG.

- **Example 2: Digital options.**

Suppose that player 1 sells (or buys) to player 2 a cash-or-nothing call option with strike K and maturity T on a risky asset whose value at time T is denoted by S_T. If the no-arbitrage transaction price is associated to an equivalent martingale measure $\mathbb{Q}$, we have

$$P_1^1 = (\mathbb{E}_{\mathbb{Q}}[\mathbf{1}_{S_T \geqslant K}] - 1)\mathbf{1}_{S_T \geqslant K} + \mathbb{E}_{\mathbb{Q}}[\mathbf{1}_{S_T \geqslant K}]\mathbf{1}_{S_T < K}$$

and the game is fair in expectation on $(\Omega, \mathcal{A}, \mathbb{Q})$.

[1]By game of chance we mean, in this paper, a game whose outcome depends on some random experiments. Contrary to what happen in game theory we don't consider strategic interactions between players.

- Example 3: Elementary market games with proportional bets[2]**.**

Let player i bet a fixed amount a of its initial wealth and win the game with probability X_0^i. Supposing that $X_0^1 + X_0^2 = 1$ (reasoning in proportion of the total initial wealth $X_0^1 + X_0^2$ instead of using absolute values) we have

$$P_1^i = a(1 - X_0^i)\mathbf{1}_{U_1 \leqslant X_0^i} - aX_0^1\mathbf{1}_{U_1 > X_0^i}$$

where U_1 follows a uniform distribution on $[0, 1]$ and the game is fair in expectation.

Remark: *The calabash games are generic in their principle in the sense that elementary market games with proportional bets may be seen as asymptotic versions of calabash games (see Appendix).*

1.2 Numerical study of elementary market games with proportional bets

We consider in this subsection a population of $N = 100$ players with uniformly distributed initial wealth: $\forall i \in \{1, ..., 100\}$, $X_0^i = 1/100$. For each stage we select randomly and independently two players that play an elementary market game with proportional bets (see Example 3) with $a = 10\%$. Even if FEG are played at each round, the randomness clearly induces disparities in wealth between economical agents.

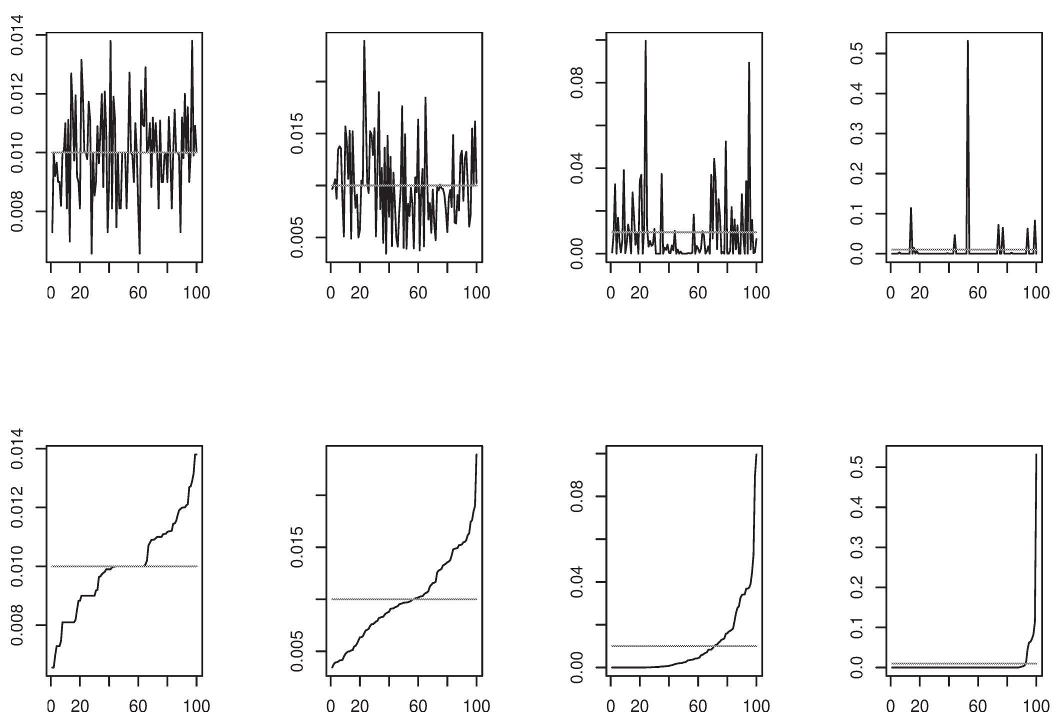

Figure 1: *The first line (resp. the second line) represents the distribution of the wealth (resp. the distribution of the increasing rearrangement of wealth) of the* $N = 100$ *players after* $n = 100$ *(first column),* $n = 1000$ *(second column),* $n = 10000$ *(third column) and* $n = 100000$ *(fourth column) FEG starting from uniformly distributed initial wealth.*

[2]It is reasonable to suppose that the amount of money bet by each economical agent is proportional to its wealth (wealthy people tend to invest more than the less wealthy). For example, this kind of mechanism is considered in [5] in the framework of econophysics.

In Figure 1, we represent the distribution of the wealth and of its increasing rearrangement after a large number of steps. We can see that inequalities become greater and greater while a poor may become richer and a rich may become poorer at each stage[3]. In particular the percentage of players that own less than the average wealth increases from around 50% after 100 transactions to around 90% after 100000: poverty traps appear, as underlined in Figure 2. To support this intuition, we represent in Figure 3 the evolution of the Gini coefficient [13] and of the Lorentz curve [18] as functions of the number of transactions. We remark without ambiguities that the repartition of wealth converges, according to these classical indicators, toward the maximal inequality case even if elementary exchange mechanisms are fair in expectation[4].

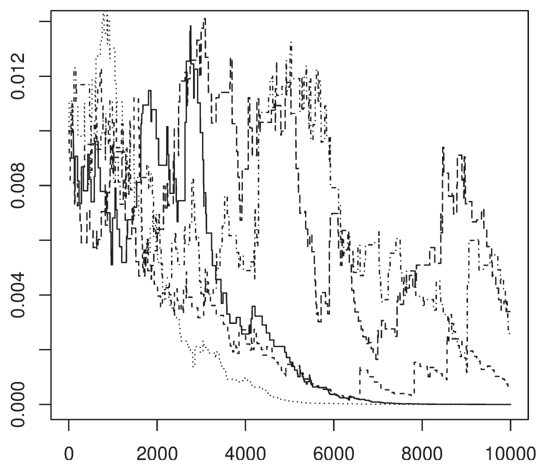

Figure 2: *Five individual wealth trajectories: creation of poverty traps*

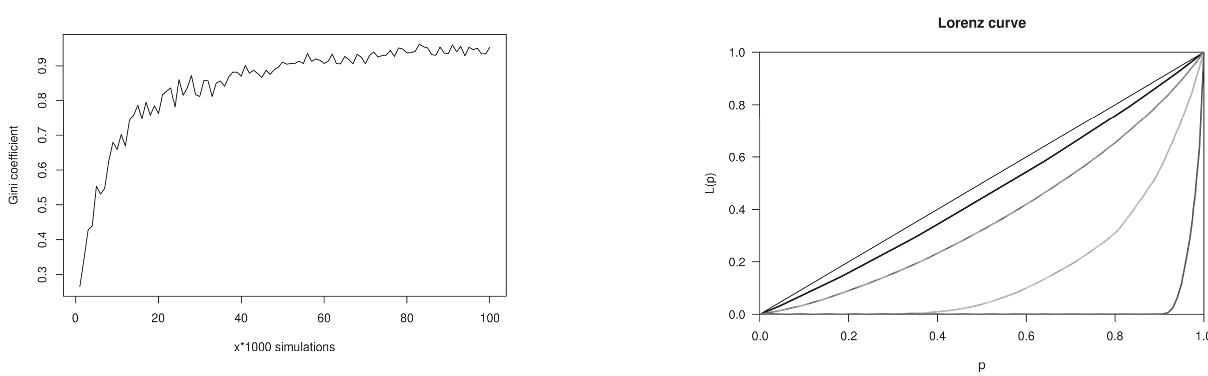

Figure 3: *Gini coefficient (left) and Lorentz curve (right). The value of the Lorentz curve $L(p)$ represents the percentage of the poorer players that possess p percent of the total wealth. For* 100, 1000, 10000 *and* 100000 *transactions, the Lorentz curve moves away further and further from the line of perfect equality. The Gini coefficient equals $1 - 2A$ where A is the integral of the Lorentz curve. A Gini coefficient of zero expresses perfect equality. A Gini coefficient of one expresses maximal inequality.*

[3]The emergence of inequalities is also numerically observed in [25] for others fair games of chance even if the convergence toward the maximal inequality situation is not reported.

[4]In simulations that are not reported here, we can also observe that the Gini coefficient of the economy in an increasing function of a: inequalities are greater in more speculative economy.

2 Mathematical study of elementary market games with proportional bets

2.1 Two players games

2.1.1 Elementary market games with proportional bets

Let us consider the repeated version of the elementary market game with proportional bets described in Example 3. If we denote by X_n^i the wealth of player i after n transactions, we have $X_n^1 + X_n^2 = 1$ (zero-sum game) and

$$X_{n+1}^i = X_n^i + a(1 - X_n^i)\mathbf{1}_{U_{n+1} \leqslant X_n^i} - aX_n^i\mathbf{1}_{U_{n+1} > X_n^i}$$

where $(U_k)_{k\in\mathbb{N}^*}$ is a sample of the uniform distribution on $[0, 1]$. The sequence $(X_n^i)_{n\in\mathbb{N}}$ is a Markov chain with

$$\mathbb{E}[X_{n+1}^i \mid X_n^i] = (X_n^i + a(1 - X_n^i))X_n^i + (X_n^i - aX_n^i)(1 - X_n^i) = X_n^i.$$

Thus $(X_n^i)_{n\in\mathbb{N}}$ is a non-negative and bounded martingale that converges almost-surely and in L^p $(1 \leqslant p < \infty)$ toward a random variable X_∞^i that is invariant with respect to the transition probability of the chain given by

$$P(x, dy) = x\varepsilon_{x+a(1-x)}(dy) + (1 - x)\varepsilon_{x-ax}(dy)$$

where $\varepsilon_\alpha(dy)$ is the Dirac mass at the point $\alpha \in \mathbb{R}$. Passing to the limit in the relation

$$\mathbb{E}[(X_{n+1}^i - X_n^i)^2 \mid X_n^i] = X_n^i a^2(1 - X_n^i)^2 + (1 - X_n^i)a^2(X_n^i)^2 = a^2X_n^i(1 - X_n^i)$$

we deduce that $X_\infty^i \in \{0, 1\}$ and from $\mathbb{E}[X_\infty^i \mid X_n^i] = X_n^i$ that X_∞^i follows a Bernoulli distribution of parameter X_0^i.

After a infinite number of transactions, one player concentrates all the wealth as empirically observed in the numerical exercise of Section 1. Nevertheless it is easy to see that it is not possible for one player to be ruined after a finite number of rounds because the random variable X_n^i ranges across the interval $[(1-a)^nX_0^i, 1+(1-a)^n(X_0^i-1)]$. If we want to obtain theoretical approximations of almost-bankruptcy times, analytic computations quickly become prohibitive. In the next part, we give a precise answer to this question at the very least in the case of small and high-frequency transactions.

Remark: *In the repeated version of the elementary calabash game described in Example 1, the situation is different, due to the fact that bets are discrete. Let X_n^i be the wealth of player i after n transactions and N_n^i the number of seeds bet by player i at stage n with $0 < N_n^i \leqslant X_n^i$ if $X_n^i \neq 0$ and $N_n^i = 0$ otherwise (the game is finished when one of the player is ruined). We have $X_n^1 + X_n^2 = N$ (zero-sum game) and*

$$X_n^1 = X_0^1 + \sum_{1 \leqslant n \leqslant k} P_j^1$$

where

$$P_n^1 = N_n^2\mathbf{1}_{U_n \leqslant \frac{N_n^1}{N_n^1+N_n^2}} - N_n^1\mathbf{1}_{U_n > \frac{N_n^1}{N_n^1+N_n^2}}$$

and where $(U_n)_{n\in\mathbb{N}^*}$ is a sample of the uniform distribution on $[0, 1]$. If we denote by $(\mathcal{F}_n)_{n\in\mathbb{N}^*}$ the filtration generated by the $(U_n)_{n\in\mathbb{N}^*}$, supposing that the processes $(N_n^i)_{n\in\mathbb{N}^*}$ and $(N_n^2)_{n\in\mathbb{N}^*}$ are predictable, the process $(X_n^1)_{n\in\mathbb{N}}$ is a bounded martingale fulfilling

$$\mathbb{E}[(X_{n+1}^1 - X_n^1)^2 \mid \mathcal{F}_n] = N_{n+1}^1 N_{n+1}^2.$$

Thus, in this case, one of the player is almost-surely ruined in finite time.

2.1.2 Elementary market games with proportional bets: Continuous time case

In this section we prove that the sequence of stochastic processes obtained from the preceding Markov chains by transforming the time scales and state spaces appropriately[5] converges weakly to a diffusion process, the latter being more amenable to analysis.

Let $f : \mathbb{R} \to \mathbb{R}$ be a measurable and bounded mapping. For all $a \in \mathbb{R}_+$ and $x \in]0,1[$ we define the generator A_a of the elementary market game with parameter a:

$$A_a[f](x) = \mathbb{E}[f(X_1^1) - f(X_0^1) \mid X_0^1 = x] = xf(x + a(1-x)) + (1-x)f(x - ax) - f(x).$$

In particular, when f is of class C^∞ with a compact support in the interval $]0,1[$, we obtain from Taylor expansion that $\frac{1}{a^2}A_a f$ uniformly converges toward $\frac{1}{2}x(1-x)f"(x)$ when a goes to 0.

Considering the process $(Z_t^a)_{t\in\mathbb{R}_+}$ that is the rescaled (at frequency a^2) continuous time linear interpolation of the sequence $(X_n^1)_{n\in\mathbb{N}}$ with $X_0^1 = x$:

$$\begin{aligned} Z^a_{na^2} &= X_n^1 \quad \forall n \geqslant 0 \\ Z^a_{(n+\theta)a^2} &= Z^a_{na^2} + \theta(Z^a_{(n+1)a^2} - Z^a_{na^2}) \quad \theta \in [0,1] \quad \forall n \geqslant 0, \end{aligned}$$

we obtain from classical arguments (see for example [26] Chap. 11) the uniform weak convergence of $(Z_t^a)_{t\in\mathbb{R}_+}$ when a goes to 0 toward the diffusion process $(X_t)_{t\in\mathbb{R}_+}$, associated to the infinitesimal generator

$$A[f](x) = \frac{1}{2}x(1-x)f"(x), \tag{1}$$

that is the unique strong solution of the Stochastic differential equation

$$dX_t = \sqrt{X_t(1-X_t)}dB_t \qquad 0 < X_0 < 1 \tag{2}$$

where B_t is a standard Brownian motion[6].

The diffusion process (2) is known in mathematical genetics as the Wright-Fisher process. It is often encountered as diffusion approximation of classical discrete stochastic models used in populations genetics (see [9] Chap. 7 or [10] Chap. 3)[7].

[5]The convergence result we obtain in this section is in some sense an answer to the following remark of Sheng in [25] p. 493: *"Then there arise some difficulties in the conversion problem, which place some restrictions on the choice of the size and number of bets."*

[6]This result of convergence requires the existence and the unicity of the martingale problem associated to the generator A that is equivalent to the weak existence and the unicity in distribution of the solution of the associated stochastic differential equation. Here, (2) having Hölderian coefficients of order $\frac{1}{2}$, from the Yamada-Watanabe theorem (see [23] p. 360) we even deduce the strong existence and unicity of the solution of (2).

[7]If we consider a fixed population of size N (representing for example genes) with individuals that can be of two different types (two alleles), the simplest neutral Wright-Fisher model of evolution assumes that generation $(k+1)$ is formed from generation k by choosing N genes at random with replacement. If we denote by Y_n^N the number of individuals of type 1 in generation n, we have

$$\mathbb{P}(Y_{n+1}^N = i | Y_n^N = j) = C_N^i \left(\frac{j}{N}\right)^i \left(1 - \frac{j}{N}\right)^{N-i}$$

and the process $X_t^{(N)} = \frac{1}{N} Y_{[tN]}^{(N)}$ weakly converges toward the Wright-Fisher diffusion (see [14]). Moreover, in this case, the fixation time corresponding to the disappearance of type 2 individuals is finite almost surely (contrary to what happens for the ruin time in elementary games with proportional bets).

The points 0 and 1 are absorbing since the constant processes 0 and 1 are solutions of (2). The process $(X_t)_{t\in\mathbb{R}_+}$ is then a continuous and uniformly integrable martingale that converges almost-surely toward 1 with probability X_0 and toward 0 with probability $(1 - X_0)$. The mapping

$$u(x) = -2[(1-x)\log(1-x) + x\log x]$$

being null at the boundary of $[0, 1]$ and fulfilling $Au = -1$ on $]0, 1[$, we obtain from the Dynkin's formula (see [8] Chap. 13) that

$$u(x) = \mathbb{E}[T \mid X_0 = x]$$

where T is the hitting time of the boundary $\{0, 1\}$. Thus, T is in L^1 and so almost-surely finite. In particular starting from $X_0 = \frac{1}{2}$, the mean hitting time is $2\log 2$.

Remark: *Using the same approach, we can even show that e^T is in L^1. In fact, the mapping $h(x) = x(1-x)$ fulfills $Ah = -h$, thus defining $\xi = h + u + 1$ we have $A\xi + h + 1 = 0$ and $\lim\xi = 1$ at the boundary of $[0, 1]$. From the proof of Th. 13.17 in [8] we deduce that*

$$\xi(x) = \mathbb{E}_x[\exp\{\int_0^T \frac{1+h(X_s)}{\xi(X_s)}\}ds].$$

The result follows because the continuous mapping $\frac{1+h(x)}{1+u(x)+h(x)}$ is bounded below (its minimum is reached at $\frac{1}{2}$ and is equal to $\frac{5}{5+8log(2)}$).

In the Markov chain case, it is not possible for one player to be ruined after a finite number of rounds. When we pass to the limit, the stepsize of the chain is shortened proportionally to a^2 that goes to zero. If for small $\varepsilon > 0$, T_ε denotes the Markov chain exit time from $[\varepsilon, 1-\varepsilon]$, when a is small enough, T_ε should[8] be of order $u(x)/a^2$ that is a decreasing function of a (the ruin time is smaller for more speculative games). This intuition is confirmed by the simulations presented in Table 1.

a	"Theoritical" ruin time	Empirical ruin time
0.1	139	144
0.08	216	218
0.05	552	529
0.03	1533	1464
0.01	13900	13294

Table 1: *Theoretical and empirical ruin times for elementary market games with proportional bets: The theoretical ruin time corresponds to the Wright-Fisher approximate $\frac{2log(2)}{a^2}$ while the empirical one is obtained as the average first exit time from $[\varepsilon, 1-\varepsilon]$ obtained for 1000 independent runs of the Markov chain game with $X_0^1 = X_0^2 = 0.5$.*

2.1.4 The buffering effect of a tax rate in the economy

In this section we study the impact of a small tax rate on the dynamic of the preceding Markov chain. One of the simplest hypothesis is to consider a proportional capital tax rate that is collected

[8] This intuition is reinforced from the following theoretical result: if T_ε^a denotes the exit time of the process Z^a from $[\varepsilon, 1-\varepsilon]$, T_ε^a converges in distribution when a goes to zero toward the corresponding diffusion exit time ([12] Problem 3, Chap. 10).

at any stage and uniformly reallocated to the players. With a tax rate b (b fulfilling $0 < a + b < 1$), the transition of the Markov chain becomes

$$X^i_{n+1} = (1-b)X^i_n + a(1-X^i_n)\mathbf{1}_{U_{n+1} \leqslant X^i_n} - aX^i_n\mathbf{1}_{U_{n+1} > X^i_n} + \frac{b}{2}$$

where $(U_k)_{k\in\mathbb{N}^*}$ is a sample of the uniform distribution on the unit interval[9]. This game remains a zero-sum game that is not a FEG in general. In fact, we deduce easily from the preceding dynamic that

$$\mathbb{E}[X^1_{n+1} - X^1_n] = -b(1-b)^n(X^1_0 - \frac{1}{2}).$$

Thus the game is FEG if and only if $X^1_0 = \frac{1}{2}$ (uniform initial wealth) when $b > 0$ and favors the poorest player at any step [10] for different initial endowments. The state space of this Markov chain being compact there exists at least one invariant distribution that is not a priori unique.[11]

We first perform a numerical study similar to the untaxed case: in a population of $N = 100$ players with uniformly distributed initial wealth: $\forall i \in \{1, ..., 100\}$, $X^i_0 = 1/100$, we select randomly and independently two players that play an elementary market game with proportional bets (see Example 3) with $a = 10\%$ and $b = 1\%$. The impact of the small tax parameter b is substantial as observed comparing Figures 1 and 4. The wealth distribution remains mostly uniform stages after stages, the effect of randomness is regulated[12]. In particular the percentage of players that own less than the average wealth stabilizes around 60%. Similarly, we have represented in Figure 5 the impact of b on the Gini coefficient of the population after $n = 100000$ transactions. Even for very small values of b the Gini coefficient reduces drastically.

In spite of its simplicity, it is a priori difficult to obtain explicitly one invariant measure of such a Markov chains when $b \neq 0$ [13]. Therefore, we study the diffusion limit of the model for small and high-frequency transactions. Following a similar approach as in Section 2.1.2, if $f : \mathbb{R} \to \mathbb{R}$ is a measurable and bounded mapping, $\forall a \in \mathbb{R}_+$ and $x \in]0, 1[$ the generator A_a of the elementary taxed market game with parameters a and b becomes

$$A_a[f](x) = \mathbb{E}[f(X^1_1) - f(X^1_0) \mid X^1_0 = x] = xf((1-b)x + a(1-x)) + (1-x)f((1-b)x - ax) - f(x).$$

In particular, when f is of class C^∞ with a compact support in the interval $]0, 1[$, we obtain from Taylor expansion that $\frac{1}{a^2}A_a f$ uniformly converges toward

$$A[f](x) = \frac{1}{2}x(1-x)f'' + \frac{\lambda}{2}(1-2x)f'$$

[9] Another natural choice is to impose the following dynamic

$$X^i_{n+1} = (1-ba)X^i_n + a(1-X^i_n)\mathbf{1}_{U_{n+1} \leqslant X^i_n} - aX^i_n\mathbf{1}_{U_{n+1} > X^i_n} + \frac{ba}{2},$$

in this case the parameter b may be interpreted as a proportional transaction cost. All the results of this section remain valid taking $b = \lambda a$ instead of $b = \lambda a^2$.

[10] Nevertheless, it may be seen as asymptotically FEG because $\mathbb{E}[X^1_{n+1} - X^1_n] \to 0$.

[11] The proof of the unicity of such a stationary distribution is not crucial for our purpose because this distribution won't be explicitly known. Moreover, any of these possible invariant distributions will be well approximated, for small and high-frequency transactions, by the unique invariant stationary distribution of the Wright-Fisher diffusion with mutations (see the proof of the Th. 2.2 of [12] p. 418).

[12] A similar mechanism of taxation is numerically studied in [3] with analogous conclusions.

[13] The same holds for the Wright-Fisher Markov chain with mutations.

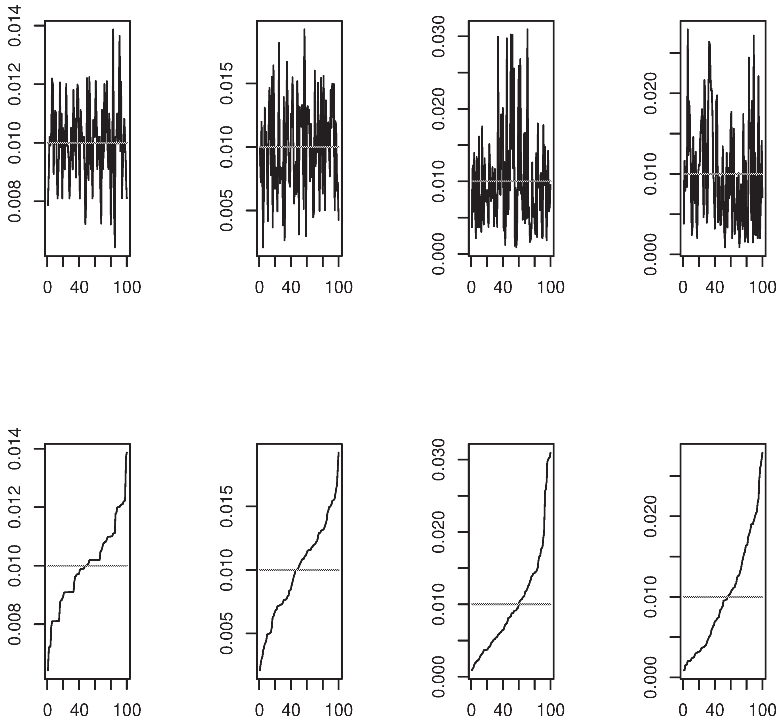

Figure 4: *The first line (resp. the second line) represents the distribution of the wealth (resp. the distribution of the increasing rearrangement of wealth) of the* $N = 100$ *players after* $n = 100$ *(first column),* $n = 1000$ *(second column),* $n = 10000$ *(third column) and* $n = 100000$ *(fourth column) FEG starting from uniformly distributed initial wealth with* $a = 0.1$ *and* $b = 0.01$*.*

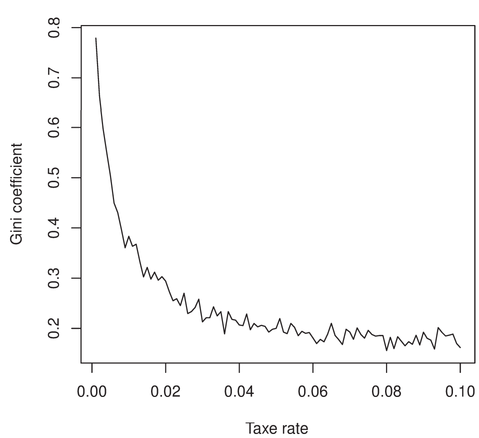

Figure 5: *Dependence of the Gini coefficient on the tax rate* b *in a population of* $N = 100$ *players after* $n = 100000$ *transactions.*

when a goes to 0 and $b = \lambda a^2$. The infinitesimal generator A is associated to the diffusion

$$dX_t = \sqrt{X_t(1 - X_t)}dB_t + \frac{\lambda}{2}(1 - 2X_t)dt \qquad 0 < X_0 < 1 \tag{3}$$

where B_t is a standard Brownian motion. This diffusion process is classically known (see [9] Chap. 7.2) as the one dimensional Wright-Fisher diffusion with mutations, the mutation rates being identical (equal to $\frac{\lambda}{2}$) for the two alleles[14].

Remark: *We see on our model that a tax on the income yields a completely different asymptotic result from a tax on the owned capital. In fact, in the Markov chain game, the rate b (and so λ) may be seen as a capital tax rate because a fixed proportion of the wealth is collected. If we consider instead an income tax uniformly redistributed among players, we obtain the following FEG*

$$X^i_{n+1} = X^i_n + a(1-\frac{b}{2})(1-X^i_n)\mathbf{1}_{U_{n+1}\leqslant X^i_n} - a(1-\frac{b}{2})X^i_n\mathbf{1}_{U_{n+1}>X^i_n}.$$

Here, we recover the untaxed dynamic with changed parameters, thus, when b is a constant, the diffusion limit is related to the Wright-Fisher diffusion without mutations up to the factor $(1-\frac{b}{2})^2$:

$$dX_t = (1-\frac{b}{2})^2\sqrt{X_t(1-X_t)}dB_t.$$

In particular, the parameter b is not sufficient to prevent the convergence toward the maximal inequality case and its only effect is to slow down the ruin. In fact, when T_b is the hitting time of the boundary $\{0,1\}$ for the associated diffusion, we have by analogy with the untaxed case

$$\mathbb{E}[T_b \mid X_0 = x] = -\frac{2}{(1-\frac{b}{2})^2}[(1-x)\log(1-x) + x\log x].$$

In [21] where players follow optimal consumption-bequest plans the same distinction has been pointed out between capital and income taxes with similar conclusions.

In the presence of tax ($\lambda > 0$), looking for an invariant measure m on $]0,1[$ fulfilling

$$\int_0^1 A[f](x)g(x)dm(x) = \int_0^1 A[g](x)f(x)dm(x)$$

for f and g of class C^∞ with a compact support in the interval $]0,1[$, we find a Beta probability distribution

$$m(dx) = \frac{[x(1-x)]^{(\lambda-1)}}{\beta(\lambda,\lambda)}dx$$

that is symmetric with respect to the uniform initial wealth case $x = \frac{1}{2}$.

If $\lambda > 1$, there is a strong restoring force in the direction of $x = \frac{1}{2}$ because the tax rate is high, if $\lambda = 1$, the Lebesgue measure on $[0,1]$ is invariant and if $0 < \lambda < 1$, the restoring force is partially offset, the density of the invariant measure approaches infinity near the boundaries 0 and 1 remaining of finite mass. In all these cases, the Wright-Fisher diffusion is ergodic and converges in distribution toward the invariant probability measure. In Figure 6, we have represented the invariant density functions of the wright-Fisher diffusion with mutations for different values of the tax rate λ to compare them with the empirical distribution of X^1_{100000} in the Markov chain market game with $b = \lambda a^2$. In both cases we start from a maximal inequality case ($\lambda = 0$) to be more and more concentrated around $x = \frac{1}{2}$ when λ increases.

[14]In order to obtain at the limit the Wright-Fisher diffusion with different mutation rates

$$dX_t = \sqrt{X_t(1-X_t)}dB_t + \frac{\lambda_2}{2}(1-X_t)dt - \frac{\lambda_1}{2}X_tdt$$

we simply have to consider two different tax rates $b_i = \lambda_i a^2$ for the players.

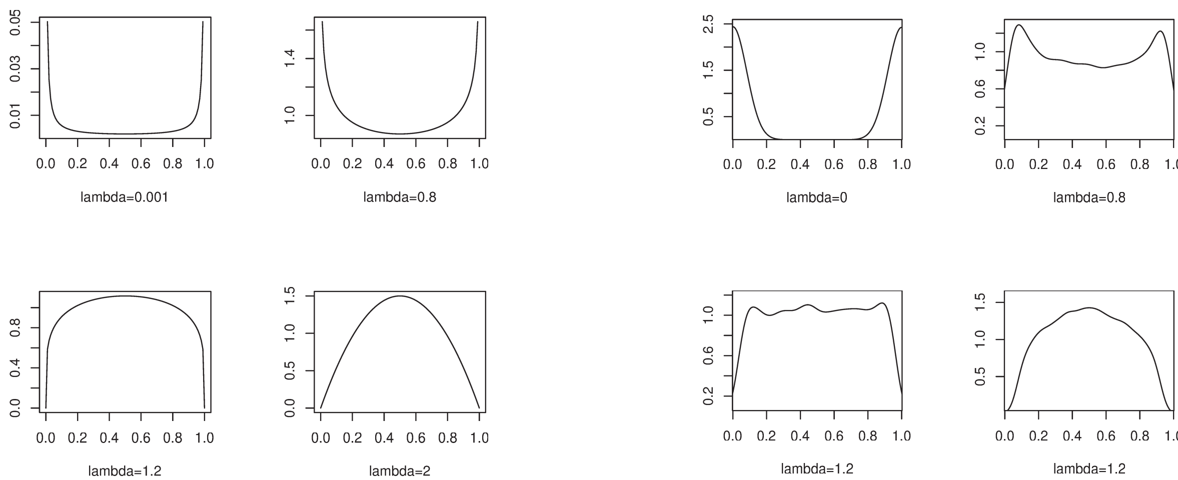

Figure 6: *Invariant density function of the Wright-Fisher diffusion with mutations for different values of λ (left part) and empirical distribution (using 1000 independent Monte Carlo simulations) of X^1_{100000} in the Markov chain market game with parameters $a = 0.1$ and $b = \lambda a^2$ (right part).*

To complete the analogy between the Markov chain market game and the limiting Beta distribution, remind that (see [22]) the Gini coefficient associated with the Beta(λ, λ) distribution is given by

$$\text{Gini}(\lambda) = \frac{2\beta(2\lambda, 2\lambda)}{\lambda\beta(\lambda, \lambda)^2}.$$

At stationarity, the small wealth probability fulfills

$$m\left(\{X^1_t \leqslant \varepsilon\} \cup \{X^2_t \leqslant \varepsilon\}\right) \underset{0^+}{\sim} \frac{2\varepsilon^\lambda}{\beta(\lambda, \lambda)\lambda} = p(\lambda).$$

When $\varepsilon < \frac{1}{4}$, from the properties of the Beta function (see [1] Chap. 6), we can prove[15] that p is a decreasing function of λ (the small wealth probability decreases when the tax rate increases) and that

$$\lim_{\lambda\to 0} p(\lambda) = 1 \text{ and } \lim_{\lambda\to\infty} p(\lambda) = 0.$$

In Table 2 (resp. Table 3) we compare, for different values of λ, the Gini coefficient (resp. small wealth probability) obtained from the Beta(λ, λ) distribution and the empirical Gini coefficient (resp. empirical small wealth probability) obtained from 1000 independent realizations of X^1_{100000} in the Markov chain market game with $b = \lambda a^2$: the results are close together.

[15] If we denote by ψ_n the polygamma function of order n (see [1] Chap. 6), we have $log(p(\lambda))' = log(\varepsilon) + g(\lambda)$ where

$$g(\lambda) = -2(\psi_0(\lambda) - \psi_0(2\lambda)) - \frac{1}{\lambda} = \psi_0(\lambda + \frac{1}{2}) - \psi_0(\lambda + 1) + 2log(2).$$

Since $g'(\lambda) = \psi_1(\lambda + \frac{1}{2}) - \psi_1(\lambda + 1)$ we can see that g is an increasing function (ψ_1 being a decreasing one) bounded by $2log(2)$. Thus, if $\varepsilon < \frac{1}{4}$, p is strictly decreasing. From $\beta(\lambda, \lambda) \underset{0^+}{\sim} \frac{2}{\lambda}$ and $\beta(\lambda, \lambda) \underset{+\infty}{\sim} \frac{2\sqrt{\pi}}{\sqrt{\lambda}}$ we deduce easily that

$$\lim_{\lambda\to 0} p(\lambda) = 1 \text{ and } \lim_{\lambda\to\infty} p(\lambda) = 0.$$

λ	Theoretical Gini coefficient	Empirical Gini coefficient
1	0.33	0.32
2	0.26	0.25
5	0.17	0.18
8	0.14	0.14
10	0.12	0.12

Table 2: *Comparison between the Gini coefficient obtained theoretically from the Beta(λ, λ) distribution and the empirical Gini coefficient obtained from* 1000 *independent realizations of* X^1_{100000} *in the Markov chain market game with* $a = 0.1$ *and* $b = \lambda a^2$.

λ	Theoretical small wealth probability	Empirical small wealth probability
0.1	0.81	0.85
0.5	0.4	0.44
1	0.2	0.23
2	0.06	0.05
3	0.02	0

Table 3: *Comparison between the small wealth probability* ($\varepsilon = 0.1$) *obtained theoretically from the Beta(λ, λ) distribution and the empirical small wealth probability obtained from* 1000 *independent realizations of* X^1_{100000} *in the Markov chain market game with* $a = 0.1$ *and* $b = \lambda a^2$.

More generally, if we are interested in the players' ruin problem (equivalent to know if the points 0 or 1 are accessible for the Wright-Fisher diffusion with mutations) remind that, for an infinitesimal generator of the form

$$A[f](x) = \frac{1}{2}x(1-x)f'' + b(x)f',$$

the scale function is given by

$$s(x) = \int_{1/2}^{x} \exp[-\int_{1/2}^{y} \frac{2b(z)}{z(1-z)}dz]dy.$$

Thus, the process being recurrent (0 or 1 are inaccessible) if and only if $s(0+) = -\infty$ and $s(1-) = +\infty$ (see [23] Ex. 3.21 p. 298), for the Wright-Fisher diffusion with mutations no players are ruined in finite time as long as $\lambda > 1$. When it is strictly greater than 1, the tax rate λ induces a diversity in the economy similar to the mutation rate impact in the genetic mixing. For the Markov chain game, this condition implies $b > a^2$. For example, when $a = 10\%$ the only condition to stabilize the economy is to impose a tax rate strictly greater than the realistic value of 1%.

To conclude this section, let us remark that the Beta distribution ([22]) and some of its generalizations (see [19] or [20]) have been widely used in the literature as descriptive models for the size distribution of income and are interesting alternatives to Pareto-like distributions. Here we find a very simple agent-based model for understanding how it can naturally appear in wealth repartition problems through fair elementary games and we see that the parameter of the obtained symmetric (about $\frac{1}{2}$) Beta distribution is related to the underlying tax rate.

2.2 N players games

The aim of this section is to extend the study of the preceding dynamics (with or without tax) to the N players case. This can be achieved via different transaction mechanisms but we suppose here that one player plays against all his opponents at each stage and we consider a proportional tax uniformly distributed among all the players. In other words, if we denote by $X_n = (X_n^1, ..., X_n^N)$ the vector of wealth after n rounds we have $\forall n \in \mathbb{N}$, $\forall i \in \{1, ..., N\}$,

$$X_{n+1} = (1-(a+b))X_n + ae_i + \frac{b}{N} \text{ with probability } X_n^i$$

with $a+b<1$ and where e_i denotes the vector with a 1 in the ith coordinate and 0's elsewhere. It is easy to see by induction that starting from a point X_0 in the simplex

$$\left\{ x = (x_1, ..., x_N) \in \mathbb{R}^N \mid \sum_{i=1}^N x_i = 1, x_i \geqslant 0 \right\},$$

$(X_n)_{n\in\mathbb{N}}$ stays in the simplex.

Now we study the asymptotic behavior of the vectorial Markov chain $(X_n)_{n\in\mathbb{N}}$ with or without tax, we present only the main lines that follow the 2 players case.

2.2.1 The case without tax, b = 0.

The vector $(X_n)_{n\in\mathbb{N}}$ is once again a bounded martingale that converges almost surely and in L^p $1 \leqslant p < +\infty$, toward X_∞ that is invariant with respect to the transition of the chain. From

$$\mathbb{E}[\mid X_{n+1} - X_n \mid^2 \mid X_n] = \sum_{i=1}^N X_n^i [\sum_{j\neq i} a^2 X_n^j + a^2(1-X_n^i)^2] = a^2(1 - \sum_{i=1}^N (X_n^i)^2)$$

we deduce that X_∞ almost surely belongs to the vertices of the simplex and that $X_\infty = e_i$ with probability X_0^i.

To study the ruin times of the players, we consider the game with small and high frequency transactions: If F is a function from $\mathbb{R}^N$ into $\mathbb{R}$ of class C^∞ with compact support we have

$$A_a[F](x) = \mathbb{E}[F(X_1) - F(X_0) \mid X_0 = (x_1, ..., x_N)] = \sum_{i=1}^N x^i F((1-a)X_0 + ae_i) - F(X_0).$$

Using the Taylor formula we prove that $\frac{1}{a^2} A_a[F](x)$ uniformly converges, when a goes to 0, toward

$$A[F](x) = \frac{1}{2}\sum_{i=1}^N x^i(1-x^i)F''_{i^2}(x) - \sum_{i<j} x^i x^j F''_{ij}(x) = \frac{1}{2}\sum_{i=1}^N x^i(1-x^i)F''_{i^2}(x) - \frac{1}{2}\sum_{i\neq j} x^i x^j F''_{ij}(x).$$

Letting $\delta_{ij} = 1$ if $i = j$ and 0 otherwise, the infinitesimal generator A becomes

$$A[F](x) = \frac{1}{2}\sum_{i,j=1}^N x^i(\delta_{ij} - x^j)F''_{ij}(x).$$

This generator is classically associated to the N-allele Wright-Fisher diffusion $(X_t)_{t\in\mathbb{R}_+}$ (see [9] Chap. 8) and the rescaled continuous time linear interpolation at frequency a^2 of the sequence $(X_n)_{n\in\mathbb{N}}$ converges in distribution toward $(X_t = (X_t^1, ..., X_t^N))_{t\in\mathbb{R}_+}$ fulfilling $\forall i \in \{1, ..., N\}$

$$dX_t^i = X_t^i\sqrt{X_t^1}dB_t^1 + X_t^i\sqrt{X_t^2}dB_t^2 + \cdots + (X_t^i - 1)\sqrt{X_t^i}dB_t^i + \cdots + X_t^i\sqrt{X_t^N}dB_t^N$$

where $(B^1, ..., B^N)$ are independent standard Brownian motions.[16]

Following [17], we remark that the mapping u_N defined on the simplex by

$$u_N(x) = -2\left[\sum_{i=1}^{N}\varphi(x^i) - \sum_{i<j}\varphi(x^i+x^j) + \sum_{i<j<k}\varphi(x^i+x^j+x^k) - \cdots \right.$$
$$\left. \cdots + (-1)^N \sum_{j_1<j_2<\cdots<j_{N-1}} \varphi(x^{j_1}+x^{j_2}+\cdots+x^{j_{N-1}})\right]$$

where $\varphi(x) = x\log x$, is zero at the faces of the simplex and fulfills $Au_N = -1$ at its interior[17]. Thus, if T_N denotes the first hitting time of the faces of the simplex (that is the time when a first player is ruined), $\mathbb{E}[T_N \mid X_0 = (x_1, ..., x_N)] = u_N(x)$. In particular, starting from the uniform situation $X_0 = (\frac{1}{N}, \ldots, \frac{1}{N})$ we have

$$\mathbb{E}[T_N \mid X_0 = (\frac{1}{N}, \ldots, \frac{1}{N})] = u_N(X_0) = 2\sum_{k=1}^{N-1}(-1)^k C_N^k \varphi(\frac{k}{N}) \tag{4}$$

and this quantity converges toward 0 when N goes to infinity[18]. In the same way, the mapping $w_N(x) = -2\sum_{i=1}^{N}(1-x^i)\log(1-x^i)$ also fulfils $Aw_N = -1$ on the simplex except for the vertices and is zero at the vertices. Thus if S_N denotes the first hitting time of the vertices (that is the time when all the players except one are ruined), S_N is almost-surely finite and we have

$$\mathbb{E}[S_N \mid X_0 = (\frac{1}{N}, \ldots, \frac{1}{N})] = -2N(1-\frac{1}{N})\log(1-\frac{1}{N}) \to 2.$$

Remark: *Using the mapping $h_N(x) = \sum_{i=1}^{N} x^i(1-x^i)$, we can prove in the spirit of Section 2.1.2 that S_N has a finite exponential moment.*

2.2.2 The case $\mathbf{b \neq 0}$

Supposing that $b = \lambda a^2 > 0$, we can prove that the generator A_a associated to the N players game with proportional bet and tax converges toward

$$A_\lambda[F](x) = \frac{1}{2}\sum_{i,j=1}^{N} x^i(\delta_{ij} - x^j)F''_{ij}(x) + \sum_{i=1}^{N}\frac{\lambda}{N}(1-Nx_i)F'_i(x)$$

that is the infinitesimal generator associated to the N-allele Wright-Fisher diffusion with a uniform mutation rate of $\frac{\lambda}{N}$ (see [9] p. 314). The unique invariant probability measure of the associated diffusion is given by the following Dirichlet distribution:

[16] Classically this convergence holds when the limit diffusion has a unique weak solution. For $N = 2$, we have seen that the result is a simple consequence of the Yamada-Watanabe theorem (see [23] p. 360) for one dimensional diffusions. For the general case, this stochastic differential equation with bounded coefficients has a weak solution (see [16], Th. 2.2, Chap. 4). For the unicity, we can do an induction reasoning on the dimension N because in the interior of the simplex the coefficients are C^1 and Lipschitz (classical conditions for strong existence and unicity) and because the faces of the simplex are absorbing sets for the diffusion with almost-surely finite hitting times (see also [11]).

[17] If $N = 2$, $u_2(x) = -2[x^1\log(x^1) + x^2\log(x^2)]$ and we recover the function u of Section 2.1.2.

[18] If $\varphi = \sum_{p\in\mathbb{Z}} a_p e^{2i\pi px}$ is the Fourier series representation of a the function φ we have $u_N(X_0) = 2\sum_{p\in\mathbb{Z}} a_p[(1 - e^{2i\pi\frac{p}{N}})^N - 1] \to -\sum_p a_p$. But φ being of bounded variations, using the Dirichlet-Jordan test ([15]), we obtain $\sum_p a_p = \varphi(0) = 0$.

$$m_N(dx) = \frac{\Gamma(2\lambda)}{\Gamma(\frac{2\lambda}{N})^N} \prod_{i=1}^{N} x_i^{(\frac{2\lambda}{N}-1)} dx,$$

in particular, the marginal distributions are Beta$\left(\frac{2\lambda}{N}, \frac{2(N-1)\lambda}{N}\right)$ distributions and the small wealth probability of any agent fulfills

$$m_N\left(\{X_t^1 \leqslant \varepsilon\}\right) \underset{0^+}{\sim} \frac{N\varepsilon^{\frac{2\lambda}{N}}}{2\beta\left(\frac{2\lambda}{N}, \frac{2(N-1)\lambda}{N}\right)\lambda} = p_N(\lambda).$$

Thus, we can prove that (see [1] Chap. 6 and Footnote 15)

$$p_N(\lambda) \underset{N\to\infty}{\to} 1, \quad \lim_{\lambda\to 0} p(\lambda) = \frac{N-1}{N} \text{ and } \lim_{\lambda\to\infty} p(\lambda) = 0.$$

.

3 Conclusion

We have shown that a simplified economy of N agents who exchange by zero-sum games fair in expectation converges in the almost sure sense to the situation where a single agent concentrates all the wealth. The mathematical study can be pushed more accurately by considering the limit diffusion process obtained for small and frequent transactions and we recover at the limit some classical models used in population genetics.

We also prove, in our framework, that the presence of a tax on the owned capital removes this convergence to the extreme inequality even for a low tax level. The economy converges in this case to a random situation which mix the respective fortunes of the agents. Surprisingly, when income taxes are considered the dynamic is drastically different: a tax on the income only slows the dynamics towards the extreme inequality.

Let us mention finally that this study can be extended thanks to other classical mathematical tools from population genetics that provide an interesting enlighting, namely the passage to an infinite population represented by a measure [19]. This gives a new economic interpretation of the so-called Fleming-Viot process ([10]) that achieves a kind of zoom on the situation where only a few agents are not yet ruined.

[19] such a representation has been used as soon as in 1982 by G. Debreu for another problem in [7] p. 125 et seq.

4 Appendix

Elementary market games with proportional bets may be seen as randomized versions of the calabash game of Example 1.

In fact, if we take in the framework of Example 1, $X_0^1(N) = [p_N N]$ where $[x]$ denotes the entire part of x and where $(p_N)_{N\in\mathbb{N}^*}$ is a sequence in $[0,1]$ that converges toward $p \in]0,1[$ and

$$N_1^i(N) = 1 + \sum_{k=1}^{X_0^1(N)-1} H_k^i$$

where the $(H_k^i)_{(i,k)\in\{1,2\}\times\mathbb{N}^*}$ are independent random variables such that H_k^i follows a Bernoulli distribution of parameter $\frac{aX_0^i(N)-1}{X_0^i(N)-1}$. Thus, we obtain a one stage calabash game where the players randomly select the number of seeds with bets that are proportional in expectation to their initial wealth because $\mathbb{E}[N_1^i(N)] = aX_0^i(N)$. From

$$X_1^1(N) = X_0^1(N) + N_1^2(N)\mathbf{1}_{U_1 \leqslant \frac{N_1^1(N)}{N_1^1(N)+N_1^2(N)}} - N_1^1(N)\mathbf{1}_{U_1 > \frac{N_1^1(N)}{N_1^1(N)+N_1^2(N)}},$$

if we suppose that the sequence $(H_k^i)_{(i,k)\in\{1,2\}\times\mathbb{N}^*}$ is independent of U_1 we have $\forall t \in \mathbb{R}$

$$\mathbb{E}\left[e^{it\frac{X_1^1(N)}{N}}\right] = e^{it\frac{X_0^1(N)}{N}}\mathbb{E}\left[e^{it\frac{N_1^2(N)}{N}}\frac{N_1^1(N)}{N_1^1(N)+N_1^2(N)} + e^{-it\frac{N_1^1(N)}{N}}\frac{N_1^2(N)}{N_1^1(N)+N_1^2(N)}\right].$$

Since

$$\sum_{k=1}^{\infty}\frac{Var[H_k^i]}{k} < \infty \text{ and } \frac{1}{N}\mathbb{E}\left[\sum_{k=1}^{N} H_k^i\right] \to a,$$

we deduce from the Kolmogorov's strong law of large numbers (see [24], Th. 2.3.10) that $\frac{1}{N}\sum_{k=1}^{N} H_k^i$ converges almost surely toward a and that $\frac{N_1^1(N)}{N}$ (resp. $\frac{N_1^2(N)}{N}$) converges almost surely toward ap (resp. $a(1-p)$). Thus, by the dominated convergence theorem

$$\mathbb{E}\left[e^{it\frac{X_1^1(N)}{N}}\right] \to e^{itp(1-a)}(1-p) + e^{it(a(1-p)+p)}p$$

and $(\frac{X_0^1(N)}{N}, \frac{X_1^1(N)}{N})$ converges in distribution toward the elementary market game with proportional bets of Example 3.

References

[1] Abramowitz, M. and Stegun, I., (1964). Handbook of mathematical functions with formulas, graphs and mathematical tables. Dover: New-York.

[2] Bachelier, L., (1900). Théorie de la Spéculation. Annales scientifiques de l'École normale supérieure, 17(3), 21-86. English translation in: The Random Character of stock market prices (P. Cootner, editor), MIT Press, 1964.

[3] Banos, A., (2010). La simulation à base d'agents en sciences sociales : une béquille pour l'esprit humain ?. Nouvelles Perspectives en Sciences Sociales, 5(2), 91-100.

[4] Bose, I. and Banerjee, S., (2005). A stochastic model of wealth distribution. In Chatterjee, A., Yarlagadda, S. and Chakrabarti, B. (eds), Econophysics of Wealth Distributions, New Economic Windows, 195-198. Springer.

[5] Bouchaud, J.P. and Mezard, M., (2000). Wealth Condensation in a Simple Model of Economy. Physica A, 282, 536-545.

[6] Bouleau, N., (1998). Financial markets and martingales: Observations on Science and Speculation. Springer.

[7] Debreu, G., (1972). Theory of Value: An Axiomatic Analysis of Economic Equilibrium. Yale University Press.

[8] Dynkin, E.B., (1965). Markov Processes, Volume 2. Springer.

[9] Durrett, R., (2008). Probability Models for DNA Sequence Evolution. Springer.

[10] Etheridge, A., (2012). Some Mathematical Models from Population Genetics. École d'Été de Probabilités de Saint-Flour XXXIX-2009. Lecture notes in Mathematics. Springer.

[11] Ethier, S.N., (1976). A class of degenerate diffusions processes occurring in population genetics. Comm. Pure. Appl. Math, 29, 483-493.

[12] Ethier, S.N. and Kurtz, T.G., (1986). Markov processes: Characterization and Convergence. Wiley.

[13] Gini, C., (1921). Measurement of inequality of income. Economic Journal, 31, 22-43.

[14] Guess, H.A., (1973). On the weak convergence of the Wright-Fisher model. Stochastic process and their applications, 1, 287-3016.

[15] Khavin, V.P. and Nikol'skij, N.K. (eds), (1991). Commutative Harmonic Analysis I. Encyclopedia of Mathematical Sciences, 15. Springer.

[16] Kunita, N. and Watanabe, S., (1982). Stochastic Differential Equation and Diffusion Processes. North Holland.

[17] Littler, R.A., (1975). Loss of variability at one locus in a finite population. Math. Bio., 25, 151-163.

[18] Lorenz, M.O., (1905). Methods of measuring the concentration of wealth. Publications of the American Statistical Association, 9(70), 209-219.

[19] McDonald, J.B., (1984). Some generalized functions for the size distribution of income. Econometrica, 52, 647-663.

[20] McDonald, J.B. and Ransom, M.R., (2008). The Generalized Beta Distribution as a Model for the Distribution of Income: Estimation of Related Measures of Inequality. In Chotikapanich, D., (ed), Modeling Income Distributions and Lorenz Curves, 147-166. Studies in Equality, Social Exclusion and Well-Being. Springer.

[21] Pestieau, P. and Possen, U.M., (1979). A Model of Wealth Distribution. Econometrica, 47(3), 761-772.

[22] Pham-Gia, T. and Turkkan, N., (1992). Determination of the beta-distribution from its Lorenz curve. Mathematical and Computer Modelling, 16(12), 73-84.

[23] Revuz D. and Yor, M., (1994). Continuous Martingales and Brownian Motion. Springer.

[24] Sen, P.K. and Singer, J.M., (1993). Large sample methods in statistics. Chapman & Hall.

[25] Sheng, C.L., (1991). A New Approach to Utilitarianism: A Unified Utilitarian Theory and Its Application to Distributive Justice. Theory and Decision Library, Volume 5, Springer.

[26] Stroock, D.W. and Varadhan, S.R.S., (1979). Multidimensional diffusion processes. Springer.